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On state-dependent implication in quantum mechanics

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Abstract. This paper studies a generalization of the *a priori* implication $E \leqslant F$, meaning of course $EF = E$, in the quantum logic $\mathscr{P}(\mathscr{H})$ of all projectors. The new definition can be applied also to new pairs of events, being based on the indistinguishability of some events when considered from the standpoint of a (fixed) state ρ. The mathematical theory of factor Boolean algebras, on which this extension is based, is clarified.

1. Introduction

In the framework of quantum mechanics (QM) a new form of Boolean (hence essentially classical) logic has emerged: the so-called consistent quantum representations of logic introduced and elaborated by Omnes (1992). This form of logic is founded on merging *three basic concepts* of QM: *events* (mathematically projectors; they are essentially the same as quantum propositions), *states* (mathematically statistical operators), and the unitary *evolution operator* of the system. Evidently, the 'quantum representations of logic' are in conceptual complexity far beyond ordinary quantum logic (Beltrametti and Cassinelli 1981, Gudder 1979).

This state of affairs serves as the motivation for revisiting *state-dependent implication* (Omnes 1988) in this article. It merges only events and states without evolution.

There is a twofold physical motivation for defining state-dependent implication (as is done in section 3):

(i) the 'consistent quantum representations of logic' mentioned utilize an elaboration of this concept;

(ii) some aspects of two-subsystem composite systems (discussed in appendix 1) imply that the mentioned definition of state-dependent implication is physically relevant from the point of view of some performable experiments. (More about this near the end of section 3.)

In the next section we give the necessary mathematical concepts of quantum logic for this study.

2. Boolean algebras and factor algebras in quantum logic

Mathematically, *quantum logic* is the set of projectors $\mathscr{P}(\mathscr{H})$ (in the given separable Hilbert space $\mathscr{H}$, the state space of the quantum system) and its special structure

(Beltrametti and Cassinelli 1981, Gudder 1979). It contains Boolean algebras, but it is not usually one itself.

A priori or absolute or state-independent *implication* is well understood mathematically: two projectors E and F stand in the order relation $E \leqslant F$, by definition, if

$$EF = E \tag{1a}$$

or equivalently

$$R(E) \subseteq R(F) \tag{1b}$$

in terms of the ranges of the projectors (that are subspaces of $\mathscr{H}$).

Absolute implication $E \leqslant F$ (as a binary relation) determines almost the entire mathematical structure of $\mathscr{P}(\mathscr{H})$ (Beltrametti and Cassinelli 1981).

The relation at issue is an *order* relation, i.e. it has the properties of reflexivity ($E \leqslant E$, $\forall E \in \mathscr{P}(\mathscr{H})$), transitivity (if $E \leqslant F$ and $F \leqslant G$, $E, F, G \in \mathscr{P}(\mathscr{H})$, then $E \leqslant G$), and antisymmetry (if $E \leqslant F$ and $F \leqslant E$, $E, F \in \mathscr{P}(\mathscr{H})$, then $E = F$). Hence, the structure $\{\mathscr{P}(\mathscr{H}), \leqslant\}$ is endowed with *upper* and *lower bounds*. (If $E \leqslant F$, then F is an upper bound of E, and E is a lower bound of F.).

Quantum logic $\mathscr{P}(\mathscr{H})$ is a *complete lattice*, i.e. any subset of $\mathscr{P}(\mathscr{H})$ has a least upper bound (l.u.b., also called supremum or join and denoted by '$\vee$') and a greatest lower bound (g.l.b., also called infimum or meet and denoted by '$\wedge$'). Moreover, $\mathscr{P}(\mathscr{H})$ is an *orthocomplemented* complete lattice, i.e. for each $E \in \mathscr{P}(\mathscr{H})$, one has $E^\perp \equiv 1 - E$, such that

$$E \leqslant F \text{ implies } F^\perp \leqslant E^\perp.$$

Actually, there is a gradual strengthening of lattices: if a set is closed with respect to finite '$\vee$' and '$\wedge$' operations (i.e. with respect to these operations in any finite subset), then we have a *lattice*. If the set is closed with respect to the two σ-operations (i.e. with respect to '$\vee$' and '$\wedge$' in all at most countably infinite subsets), then we are dealing with a *σ-lattice*. Finally, if there is no limitation on the power of the sets, then we have a complete lattice.

In terms of subspaces (ranges of projectors), the join is equivalent to 'spanning', i.e. to finding the minimal subspace containing the subspaces from the given family. The meet is the same as intersection. There is a duality between these two operations (with respect to orthocomplementation '$\perp$')

$$\forall \mathscr{A} \subseteq \mathscr{P}(\mathscr{H}) \qquad (\vee_{\mathscr{A}} E)^\perp = \wedge_{\mathscr{A}} E^\perp$$

and vice versa

$$\forall \mathscr{A} \subseteq \mathscr{P}(\mathscr{H}) \qquad (\wedge_{\mathscr{A}} E)^\perp = \vee_{\mathscr{A}} E^\perp.$$

These are the so-called rules of Morgan. They establish a *duality* of join and meet with respect to orthocomplementation.

The lattice $\mathscr{P}(\mathscr{H})$ has a maximal element 1 and a minimal element 0. One has

$$\forall E \in \mathscr{P}(\mathscr{H}) \qquad E \vee E^\perp = 1 \qquad E \wedge E^\perp = 0.$$

The lattice $\mathscr{P}(\mathscr{H})$ has a number of other properties. They will not be made use of in this study.

If one thinks of the structure of $\mathscr{P}(\mathscr{H})$ in terms of the operations '$\vee$', '$\wedge$' and '$\perp$', they determine, in their turn, the absolute implication relation '$\leqslant$', because $E \leqslant F$ if and only if $E \wedge F = E$, and if and only if $E \vee F = F$.

Three elements E, F, G of $\mathscr{P}(\mathscr{H})$ form a *distributive triple* if the equalities

$$E\wedge(F\vee G)=(E\wedge F)\vee(E\wedge G) \qquad E\vee(F\wedge G)=(E\vee F)\wedge(E\vee G)$$

hold, together with the other four equalities obtained by cyclical permutation of E, F, G. If and only if every one of the triples of a given sublattice of $\mathscr{P}(\mathscr{H})$ is distributive, one speaks of a *distributive sublattice*.

If a sublattice of $\mathscr{P}(\mathscr{H})$ is distributive and orthocomplemented (i.e. closed with respect to '$\perp$'), then it is called a *Boolean subalgebra*. On account of the mentioned gradation of lattices, one speaks also of *σ-Boolean subalgebras* and of *complete Boolean subalgebras*.

Every two elements E, F of a Boolean subalgebra $\mathscr{B}$ are *compatible* or commutative:

$$EF=FE.$$

In terms of the operations this means that

$$\exists G, H, I\in\mathscr{B} \qquad H=E\wedge F \qquad E=G\vee H \qquad F=H\vee I$$

and the projectors G, H, I are *orthogonal* to each other (E is orthogonal to F if $E\leqslant F^{\perp}$). In case of orthogonality one can replace '$\vee$' by '$+$', and '$\wedge$' by multiplication.

Conversely, every compatible subset of $\mathscr{P}(\mathscr{H})$ (i.e. every subset, every two elements of which are compatible) is a subset of a Boolean subalgebra, or, as one puts it, is *covered* by one of the latter. (cf theorem 3.10 on p 49 in Gudder 1979). Since the intersection of any family of Boolean subalgebras is a Boolean subalgebra, there is a unique minimal Boolean subalgebra that covers a given commutative set of projectors (the latter 'spans' the former, as one can say). All analogous claims hold for σ-Boolean subalgebras and complete Boolean subalgebras.

The fact that any commutative set of projectors is covered by a Boolean subalgebra makes it clear that $\mathscr{P}(\mathscr{H})$ can be viewed as consisting of Boolean algebras. Since commutativity is non-transitive, for a given projector E there may exist two other projectors F and G such that they both commute with E, but do not commute with each other. Then $\{E, F\}$ and $\{E, G\}$ span two *distinct* Boolean subalgebras. Thus, any projector belongs to non-denumerably infinitely many distinct Boolean subalgebras of $\mathscr{P}(\mathscr{H})$.

One can view $\mathscr{P}(\mathscr{H})$ as obtained by '*pasting*' *Boolean algebras on top of one another*. By this one means embedding (i.e. injecting isomorphically) a Boolean subalgebra of one such algebra into another of them (to define the 'intersection' of the two Boolean algebras), so that the union of the two, though no longer a commutative set, is part of $\mathscr{P}(\mathscr{H})$.

Every complete Boolean algebra is necessarily a σ-Boolean algebra and a Boolean algebra. Every σ-Boolean algebra is a Boolean algebra. In general, there are Boolean algebras that are not σ-Boolean ones, and there are σ-Boolean algebras that are not complete ones. But *separability* of the Hilbert space $\mathscr{H}$ implies that there are *no σ-Boolean algebras* in the quantum logic $\mathscr{P}(\mathscr{H})$ that are not complete Boolean algebras. (We note that the Hilbert space of ordinary quantum mechanics is indeed separable.)

The last claim follows from the fact that in every non-denumerable set $\{E_m: m\in M\}$ ($\subset\mathscr{P}(\mathscr{H})$) ('$M$' being an index set) there exists an at most denumerable subset $\{E_1, E_2, \ldots\}\subset\{E_m: m\in M\}$ such that

$$\bigwedge_{n=1}^{\infty} E_n=\bigwedge_{m\in M} E_m$$

(see proposition 3 in Herbut 1984).

We turn now to *factor Boolean algebras*. Since group theory is better known to theoretical physicists than the theory of Boolean algebras, it might be useful to point out that a factor Boolean algebra is the counterpart of a factor group; an ideal is the analogue of an invariant subgroup.

An *ideal* Δ is a non-empty subset of a Boolean algebra $\mathscr{B}$ that has two properties (Sikorski 1964):

(i) if $E, F \in \Delta$, then also $(E \vee F) \in \Delta$;

(ii) if $E \in \mathscr{B}$, $F \in \Delta$, and $E \leqslant F$, then also $E \in \Delta$.

On the other hand, as is well known, an *equivalence relation* satisfies, besides reflexivity and transitivity, also the symmetry requirement. (In $\mathscr{P}(\mathscr{H})$, for example, this means that if E and F are equivalent, then so are F and E.) Any equivalence relation breaks up the set in which it is defined into *classes* (non-overlapping subsets, the union of which is the entire set). The set of classes is called the *quotient set*, and is denoted as the set divided by the equivalence relation.

Every ideal Δ in $\mathscr{B}$ defines an *equivalence relation* $\sim_\Delta$, as follows (Sikorski 1964):

$$E \sim_\Delta F \quad \text{if both } (E \wedge F^\perp) \in \Delta \quad \text{and} \quad (E^\perp \wedge F) \in \Delta. \tag{2}$$

The quotient set $\mathscr{B}/\sim_\Delta$ is a Boolean algebra if one defines the operations in it via arbitrary class representatives. In other words, denoting by $[E]$ the equivalence class (element of the quotient set) to which E ($\in \mathscr{B}$) belongs, one defines for all $E, F \in \mathscr{B}$:

$$[E] \vee [F] \equiv [E \vee F] \qquad [E] \wedge [F] \equiv [E \wedge F] = [EF] \qquad [E]^\perp \equiv [E^\perp]. \tag{3}$$

Note that E is an arbitrary element of the class $[E]$, and that the structure of the quotient set is 'inherited' from the elements (in the classes). It is easy to see that the operations in $\mathscr{B}/\sim_\Delta$ are well defined (i.e. consistently defined) by (3).

The Boolean algebra $\mathscr{B}/\sim_\Delta$ is called the *factor Boolean algebra* corresponding to the ideal Δ. (Similarly, a factor group corresponds to a given invariant subgroup.) It is denoted by $\mathscr{B}/\Delta$.

One should note also that Boolean implication '$\leqslant$', 'inherited' by a Boolean factor algebra $\mathscr{B}/\Delta$ from the absolute implication in the initially given Boolean algebra $\mathscr{B}$, does not go via an arbitrary class representative (in contrast to the Boolean operations). More precisely, one has:

Lemma 1. If $E \leqslant F$, then $[E] \leqslant [F]$. Conversely, if $[E] \leqslant [F]$, then $\exists$: $E' \in [E]$ and $F' \in [F]$ such that $E' \leqslant F'$.

Proof. If $E \leqslant F$, then $EF = E$, and $[E] \wedge [F] \equiv [EF] = [E]$. Hence, $[E] \leqslant [F]$. If $[E] \leqslant [F]$, then $[E] \wedge [F] = [E]$. Further, $[EF] = [E]$. Thus, putting $E' \equiv EF$, $F' \equiv F$, one has $E'F' = E'$, i.e. $E' \leqslant F'$. □

3. The definition of state-dependent implication

We begin to investigate physically and mathematically *state-dependent implication* $E \leqslant_\rho F$ of an event F by an event E in an arbitrary given state ρ. Tentatively, one may define this relation by

$$\text{Tr}\, F(E\rho E/\text{Tr}\, E\rho) = 1 \tag{4}$$

both for compatible and incompatible events E and F (commutative and non-commutative projectors) if one excludes the set

$$\mathscr{P}_0 \equiv \{E: \text{Tr}\, E\rho = 0\} \quad (\subset \mathscr{P}(\mathscr{H}))$$

of events that are 'impossible' in the state ρ.

One may interpret (4) physically as follows:

The event E *occurs* in the state ρ *in an ideal way*, i.e. by the Lüders selective change of state $\rho \rightarrow (E\rho E/\text{Tr}\, E\rho)$ (Lüders 1951, Messiah 1961, p 333). Immediately after this, the event F is measured in some way (no evolution is allowed to take place). Relation (4) then requires the event F to be certain ('statistically', i.e. in the state $(E\rho E/\text{Tr}\, E\rho)$, which determines the statistics at issue).

Relation (4) is not necessarily transitive, i.e. one may have $E \leqslant_\rho F$ and $F \leqslant_\rho G$ without having $E \leqslant_\rho G$. This is illustrated (making use of incompatible events) in appendix 2.

If one confines oneself to making use of (4) within a given Boolean subalgebra $\mathscr{B}$ of events (any two events are compatible in it), then, as proved in appendix 3, one does have transitivity. One has only to achieve reflexivity, and then relation (4) is a preorder (in $\mathscr{B} \backslash (\mathscr{B} \cap \mathscr{P}_0)$) (Birkhoff 1940, see explanation below).

Restriction to a Boolean subalgebra $\mathscr{B}$ in using (4) is natural in quantum logic because every quantum-mechanical observable A (Hermitian operator in $\mathscr{H}$), when viewed as a given spectral measure $E_A: \mathscr{B}_R \rightarrow \mathscr{P}(\mathscr{H})$, i.e. as a σ-homomorphism of the σ-Boolean algebra $\mathscr{B}_R$ (called the σ-field of Borel sets B_R on the real axis $\boldsymbol{R}$) into quantum logic $\mathscr{P}(\mathscr{H})$, has a range

$$\{E_A(B_R): B_R \in \mathscr{B}_R\} \quad (\subset \mathscr{P}(\mathscr{H}))$$

that is necessarily a σ-Boolean subalgebra (and hence a Boolean subalgebra) in $\mathscr{P}(\mathscr{H})$. And vice versa, for every given σ-Boolean subalgebra of $\mathscr{P}(\mathscr{H})$, there exists a (non-empty) set of observables each having the given σ-Boolean subalgebra as the range of its spectral measure. (Any two observables in this set are non-singular functions of each other.)

On account of commutation under the trace, the assumed commutativity of E and F in the given Boolean algebra $\mathscr{B}$, and idempotency, one can rewrite (4) (equivalently) as follows:

$$\text{Tr}\, EF\rho = \text{Tr}\, E\rho \tag{5}$$

for E for which $\text{Tr}\, E\rho > 0$, i.e. on $(\mathscr{B} \backslash (\mathscr{B} \cap \mathscr{P}_0))$.

If $\text{Tr}\, E\rho = 0$, then necessarily also $\text{Tr}\, EF\rho = 0$. This is an immediate consequence of the fact that $\text{Tr}\, E\rho = 0$ holds if and only if $E\rho = 0$ (proved in appendix 4), and of the commutation of E and F. Thus, (5) without restriction on the choice of E, i.e. with validity on the entire $\mathscr{B}$, is equivalent to (4). (More precisely, on $(\mathscr{B} \cap \mathscr{P}_0)$ (5) is an identity, whereas on $(\mathscr{B} \backslash (\mathscr{B} \cap \mathscr{P}_0))$ it is equivalent to (4).)

In order to extend our tentative definition (4) to the entire given Boolean subalgebra $\mathscr{B}$, we complete our tentative definition as follows.

Definition 1. Let $\mathscr{B}$ be a given Boolean subalgebra of quantum logic $\mathscr{P}(\mathscr{H})$, and let ρ be a given quantum state (statistical operator). Then two events (projectors) $E, F \in \mathscr{B}$

stand in the binary relation of *state-dependent implication*, which we write as $E \leqslant_\rho F$, if (5) is valid.

Thus, we have $E \leqslant_\rho F$ if the probability of E is the same as that of the coincidence event EF. One should note that if $\mathrm{Tr}\, E\rho = 0$, i.e. if the event E is 'statistically impossible' in the state ρ, then for *every* $F \in \mathscr{B}$, we have $E \leqslant_\rho F$.

The binary relation given by our definition has both the property of *reflexivity* and that of *transitivity*. As mentioned, it establishes a *preorder* (Birkhoff 1940) between E and F. This relation is a generalization of an order relation.

Unlike an order relation, a preorder like state-dependent implication '$\leqslant_\rho$' does not have the property of antisymmetry. In our case this means that there may exist two distinct events E and F such that $E \leqslant_\rho F$ and $F \leqslant_\rho E$. Thus, on account of the confinement to the fixed state ρ, such events 'imply' each other, and one cannot physically distinguish them.

What is more, the entire study in this article will show that *physical indistinguishability* of some events is at the heart of state-dependent implication.

Mathematically, a preorder (like '$\leqslant_\rho$') always determines an equivalence relation (we write it as '$\sim_\rho$' in our case) in the following way:

Definition 2. Let $\mathscr{B}$ and ρ be given. Two events $E, F \in \mathscr{B}$ are in the state-dependent *equivalence relation*, i.e. $E \sim_\rho F$, if

$$\text{both} \quad E \leqslant_\rho F \quad \text{and} \quad F \leqslant_\rho E \tag{6}$$

hold.

The equivalence relation '$\sim_\rho$' gives rise to the quotient set $\mathscr{B}/\sim_\rho$.

If '$\rightarrow$' is an arbitrary given preorder in $\mathscr{B}$, and '$\sim\rightarrow$' is the equivalence relation induced by it (cf definition 2 as an example), then the order '$\rightarrow$' induced in the quotient set $\mathscr{B}/\sim\rightarrow$ goes via an arbitrary class representative. More precisely, $[E] \rightarrow [F]$ if and only if $E \rightarrow F$. This is so also in the special case '$\rightarrow$' $\equiv$ '$\leqslant_\rho$'.

One should note that this is quite different in the case of an arbitrary factor Boolean algebra $\mathscr{B}/\Delta$, in which an order relation is induced by the absolute implication '$\leqslant$' in $\mathscr{B}$ (cf lemma 1 above).

One should note that if one defines

$$\mathscr{B}_0 \equiv \{E : E \in \mathscr{B}, \mathrm{Tr}\, \rho E = 0\} = \mathscr{B} \cap \mathscr{P}_0$$

and symmetrically

$$\mathscr{B}_1 \equiv \{E : E \in \mathscr{B}, \mathrm{Tr}\, E\rho = 1\}$$

then $\mathscr{B}_0$ *and* $\mathscr{B}_1$ *are equivalence classes with respect to* '$\sim_\rho$'. (*This is proved in appendix* 5.) The following lemma clarifies the relation between absolute implication and state-dependent implication.

Lemma 2. The absolute implication $E \leqslant F$ is valid if and only if $E \leqslant_\rho F$ holds true in *every* quantum state ρ.

Proof. That $E \leqslant F$ implies $E \leqslant_\rho F$ for every ρ is obvious from (1*a*) and definition 1. The converse claim is seen to be valid by taking $\rho \equiv |\psi\rangle\langle\psi|$. Then $\mathrm{Tr}\, EF\rho = \mathrm{Tr}\, E\rho$ boils down to

$$\langle\psi|EF|\psi\rangle=\langle\psi|E|\psi\rangle$$

and this is valid for every $|\psi\rangle\in\mathscr{H}$, $\langle\psi|\psi\rangle=1$. Then, as is well known, we must have $EF=E$. □

That state-dependent implication is actually a weakening of the absolute one is clear from the following evident corollary.

Corollary 1. If $E\leqslant F$, and ρ is an arbitrary given state, then also $E\leqslant_\rho F$.

One may rightly object to our initial definition (4) of state-dependent implication that its physical interpretation is, at first glance, restricted to ideal occurrence of the event E in the state ρ. This is a high idealization, almost impossible to achieve in the laboratory. Thus, the concept at issue seems to be threatened by lack of physical motivation for its study.

Nevertheless, as was mentioned in the introduction, there is a special case when a clear and realistic motivation can be established: the case of composite systems (see appendix 1).

The important thing to notice is that state-dependent implication holds in this case for *any kind of measurement*: ideal, repeatable non-ideal and non-repeatable (Busch *et al* 1991), in which the occurrences of the subsystem events $(E_1\otimes 1)$ and of $(1\otimes F_2)$ in a state ρ_{12} of the composite system take place (cf appendix 1). Since realizability (in the laboratory) increases as we move from left to right along the mentioned 3-tuple of possible (individual-system) measurements (they are less and less idealized), in this case state-dependent implication is experimentally checkable.

As was pointed out, state-dependent implication '$\leqslant_\rho$', being a *preorder* in a given Boolean subalgebra $\mathscr{B}$ of the quantum logic $\mathscr{P}(\mathscr{H})$ of the system, defines an *equivalence relation* '$\sim_\rho$' in $\mathscr{B}$ that breaks up the latter into classes of *physically indistinguishable* events (in the *a priori* given state ρ).

There is only one way that a preorder in $\mathscr{B}$ can be a kind of *implication*: if it '*inherits*' this from *absolute implication* (that is defined in a state-independent way in $\mathscr{P}(\mathscr{H})$). By this I mean that the state-dependent implication $E\leqslant_\rho F$ should be due to the fact that in the equivalence classes of E and of F there exist events E' and F', respectively, such that $E'\leqslant F'$ (implication in the absolute sense). Then the state-dependent 'implication' of F by E comes from the fact that F' is implied by E', but neither is E distinguishable from E' in ρ, nor is F distinguishable from F' in this state.

By a suitable mathematical theorem we show in the next section that this simple-minded and rough idea is the right intuitive basis for state-dependent implication.

The mathematical realization of this 'inheritance' goes as follows: the quotient set $\mathscr{B}/\sim_\rho$ is a *factor Boolean algebra* when the structure of $\mathscr{B}$ is naturally (i.e. via an arbitrary representative) transferred into the quotient set (Sikorski 1964).

4. Mathematical investigation of state-dependent implication

We now prove that every statistical operator ρ in $\mathscr{H}$ defines a factor Boolean algebra $\mathscr{B}/\sim_\rho$ in any given Boolean subalgebra $\mathscr{B}$ of quantum logic $\mathscr{P}(\mathscr{H})$ via the state-dependent implication '$\leqslant_\rho$' or rather the equivalence relation '$\sim_\rho$' it gives rise to.

Theorem 1. Let $\mathscr{B}$ be an arbitrary given Boolean subalgebra of the quantum logic $\mathscr{P}(\mathscr{H})$ of a separable Hilbert space $\mathscr{H}$, and let ρ be an arbitrary given statistical operator in $\mathscr{H}$. Let, finally, '$\sim_\rho$' be the equivalence relation in $\mathscr{B}$ given rise to by ρ (cf definitions 2 and 1). Then the equivalence class [0] to which the zero projector belongs ($[0]=\mathscr{B}_0$, cf section 3) is an ideal Δ in $\mathscr{B}$. Further, the equivalence relation defined by the latter (in the sense of (2)) *coincides* with '$\sim_\rho$'.

Proof.
(a) Let Q_0 be the null-projector (the one projecting onto the null space) of ρ. It is shown in appendix 6 that

$$[0]=\mathscr{B}_0=\{E\colon E\leqslant Q_0\}. \tag{7}$$

If $E, F\in\mathscr{B}$, $E\leqslant Q_0$, $F\leqslant E$, then, on account of transitivity, also $F\leqslant Q_0$. If $E_1, E_2\in\mathscr{B}_0$, and $F\equiv E_1\vee E_2$, then, since Q_0 is a common upper bound of $\{E_1, E_2\}$, it is an upper bound also of the least upper bound F. Hence, $F\leqslant Q_0$. Thus, [0] ($\equiv\Delta$) is an ideal, as claimed.

(b) Let $E, F\in\mathscr{B}$, and $E\sim_\Delta F$. Then $(E-EF)$, $(F-EF)\in\Delta$. Hence, one can write

$$\mathrm{Tr}(E-EF)\rho=0=\mathrm{Tr}(F-EF)\rho$$

implying

$$\mathrm{Tr}\,E\rho=\mathrm{Tr}\,EF\rho=\mathrm{Tr}\,F\rho$$

or, in view of definitions 1 and 2,

$$E\leqslant_\rho F\leqslant_\rho E \qquad \text{i.e.} \quad E\sim_\rho F.$$

(c) Let $E, F\in\mathscr{B}$, and $E\sim_\rho F$. Then, by definition 2, we have $E\leqslant_\rho F\leqslant_\rho E$, and this, in turn, by definition 1, means that $\mathrm{Tr}\,E\rho=\mathrm{Tr}\,EF\rho=\mathrm{Tr}\,F\rho$. But then $(E-EF)$, $(F-EF)\in\Delta$, and thus we derive $E\sim_\Delta F$. □

Corollary 2. The factor Boolean algebra $\mathscr{B}/\Delta$ is *non-trivial*, i.e. $\mathscr{B}/\Delta\neq\mathscr{B}$, if and only if ρ is *singular*.

Proof. We have non-triviality if and only if Δ contains at least one non-zero projector. It is obvious from (7) that this is the case if and only if the null-projector Q_0 of ρ is non-zero. □

Thus, if we have a *non-singular* state ρ, the state-dependent implication '$\leqslant_\rho$' it gives rise to is actually the same as absolute implication '$\leqslant$' that was *a priori* present in $\mathscr{B}$.

Theorem 2. The order '$\leqslant_\rho$' that the preorder '$\leqslant_\rho$' (determined by a given statistical operator ρ in $\mathscr{B}$, cf definition 1) induces in $\mathscr{B}/\sim_\rho$ *coincides* with the (Boolean) implication '$\leqslant$' in the Boolean algebra $\mathscr{B}/\Delta$ ($=\mathscr{B}/\sim_\rho$, cf theorem 1).

Proof.
(a) Let $E, F\in\mathscr{B}$, and $[E]\leqslant_\rho[F]$. Then $E\leqslant_\rho F$, implying

$$\mathrm{Tr}\,E\rho=\mathrm{Tr}\,EF\rho \tag{8}$$

(see definition 1 and (5)). But then also

$$\mathrm{Tr}\, E\rho = \mathrm{Tr}\, E(EF)\rho \qquad \text{and} \qquad \mathrm{Tr}\, EF\rho = \mathrm{Tr}\, (EF)E\rho$$

which mean that $E \leqslant_\rho EF$, and $EF \leqslant_\rho E$, i.e. $E \sim_\rho EF$. Now,

$$[E] \wedge [F] \equiv [E \wedge F] = [EF] = [E]$$

which means that $[E] \leqslant [F]$.

(b) Let $E, F \in \mathscr{B}$, and $[E] \leqslant [F]$. Then $[E] \wedge [F] = [E]$, i.e. $[EF] = [E]$. This means that $EF \sim_\rho E$, implying $E \leqslant_\rho EF$, i.e. $\mathrm{Tr}\, E = \mathrm{Tr}\, E(EF) = \mathrm{Tr}\, EF$. Hence, $E \leqslant_\rho F$, and $[E] \leqslant_\rho [F]$. □

If we have two states ρ and ρ' such that the null-projector of the former implies that of the latter, i.e. if

$$Q_0 \leqslant Q_0'$$

then, as seen from (7),

$$\Delta \subseteq \Delta'$$

and, as is easily shown, the factor algebra $\mathscr{B}/\Delta'$ can also be obtained from $\mathscr{B}/\Delta$ because the classes of the former consist of classes of the latter. (The classes corresponding to Δ' are larger than those corresponding to Δ.)

We obtain the *largest* classes when ρ is a *pure state*: $\rho \equiv |\psi\rangle\langle\psi|$, because then $Q_0 = I - |\psi\rangle\langle\psi|$. This, of course, implies that we have the largest family of pairs of projectors $E, F \in \mathscr{B}$ such that

$$E \leqslant_\rho F \quad \text{though NOT } E \leqslant F$$

i.e. pairs of new (state-dependent) implications.

Corollary 3. If $E \sim_\rho F$, then there exist $E' \in [E]$, and $F' \in [F]$ such that $E' \leqslant F'$.

Proof. We put $E' \equiv EF$, and $F' \equiv F$. In the proof of theorem 2, part (a), it was shown that $E' \sim_\rho E$. Finally, $EF \leqslant F$ is obvious. □

Since we have proved in theorem 1 that the two equivalence relations '$\sim_\rho$' and '$\sim_\Delta$' coincide, corollary 3 can be viewed as an immediate consequence of lemma 2.

Thus, *state-dependent implication is 'inherited' from absolute implication* on account of the contraction of events into classes of indistinguishable events (with respect to ρ) determined by the equivalence relation $\sim_\rho$ (cf definition 2).

5. A gain in methodology

We have accomplished our programme, having shown that state-dependent implication is 'inherited' from absolute implication due to indistinguishability in ρ via the mathematical 'mechanism' of a suitable factor Boolean algebra.

Nevertheless, a methodological question lingers on. Namely, it is evident that any preorder like '$\leqslant_\rho$' in $\mathscr{B}$ *consists* of the following two concepts: an equivalence relation like '$\sim_\rho$' in $\mathscr{B}$, and an order like '$\leqslant_\rho$' in $\mathscr{B}/\sim_\rho$. (By 'consists' I mean not only that the preorder induces the equivalence relation and the order, but also, conversely, an equivalence relation and an order in the quotient set determine, in their turn, a preorder in the initial set, of course, the one that determines them 'back'.)

The question arises if one can decide, *without decomposing* a given preorder in $\mathscr{B}$ into equivalence relation plus order, whether it actually amounts to the Boolean implication in a factor Boolean algebra $\mathscr{B}/\Delta$.

An affirmative answer to this question is known in the mathematical literature for the general case of classical probability theory (Omnes 1988, Rényi 1970). We re-derive it here for the special case of a Boolean algebra of projectors in $\mathscr{H}$, and utilizing, as we have so far, the structure in $\mathscr{P}(\mathscr{H})$. Our aim is, of course, an attempt at completeness of presentation.

Definition 3. Let $\mathscr{B}$ be a Boolean subalgebra of quantum logic $\mathscr{P}(\mathscr{H})$, and let '$\leqslant$' be the absolute implication in it. We call a *preorder* '$\rightarrow$' in $\mathscr{B}$ an '*implication*' (meaning by this an extension of absolute implication '$\leqslant$' to possible new pairs of projectors) if the induced equivalence relation '$\sim_{\rightarrow}$' (cf definition 2 *mutatis mutandis*) makes the equivalence class [0] of the zero projector $P \equiv 0$ an ideal Δ, if $\sim_{\rightarrow} = \sim_{\Delta}$, and if the induced order '$\rightarrow$' in $\mathscr{B}/\sim_{\rightarrow}$ amounts to the absolute implication in $\mathscr{B}/\Delta$ $(=\mathscr{B}/\sim_{\rightarrow})$.

Theorem 3. A preorder '$\rightarrow$' in $\mathscr{B}$ is an *implication if and only if* it satisfies the following three relations in $\mathscr{B}$:

(i) Whenever $E \leqslant F$ holds, so does $E \rightarrow F$.
(ii) If $E \rightarrow F$, then $F^{\perp} \rightarrow E^{\perp}$.
(iii) If $E \rightarrow F$ and $E \rightarrow G$, then $E \rightarrow (F \wedge G) = FG$.

Proof. Given in appendix 7.

Lemma 3. If (ii) is valid, then (iii) is *equivalent* to:
(iii)′ If $E \rightarrow G$, $F \rightarrow G$, then $(E \vee F) \rightarrow G$.

Proof. The claims follow immediately from the Morgan rules, i.e. from the duality of meet and join with respect to orthocomplementation in a Boolean algebra. □

Theorem 4. Let $\mathscr{B}$ be an arbitrary Boolean subalgebra of $\mathscr{P}(\mathscr{H})$, and ρ an arbitrary statistical operator. Then '$\leqslant_{\rho}$' (cf definition 1) is an *implication.*

Proof. Given in appendix 8.

The proof of theorem 3 turns out to be lengthy, that of theorem 4 rather short. Nevertheless, there is a methodological gain: the laborious proof establishes a very general result (theorem 3), which is, so to say, a part of classical Boolean algebra theory. Hence, methodologically, the right starting point of our physical investigation actually should have been whether the given preorder '$\leqslant_{\rho}$' in $\mathscr{B}$ does, or does not, satisfy the criterion for an implication (given in theorem 3). The answer is easily obtained.

Appendix 1. Two-subsystem composite systems

Let $\mathscr{H}_{12} \equiv \mathscr{H}_1 \otimes \mathscr{H}_2$ be the state space of an arbitrary two-subsystem composite system (e.g. of a two-particle system, or of one particle with orbital and spin degrees of freedom, etc). Let, furthermore, ρ_{12} be an arbitrary (mixed or pure) state (statistical operator in $\mathscr{H}_{12}$). Finally, let E_1 and F_2 be subsystem events (projectors in $\mathscr{H}_1$ and in $\mathscr{H}_2$, respectively).

Just as in the general case of compatible events, the coincidence event $(E_1 \otimes F_2)$ need not be interpreted as simultaneous occurrence of the two events, it can also be interpreted as occurrence in (immediate) succession: first $(E_1 \otimes 1)$, then, immediately after, $(1 \otimes F_2)$. For this interpretation we need a predictable change of state in the occurrence of the former event, and we are back at ideal occurrence and relation (4). However, there is a difference.

Now one can write, after multiplication with and division by the expression $\{\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12}\}$ (which is required to be positive, cf (4)):

$$\mathrm{Tr}_{12}(E_1 \otimes F_2)\rho_{12} = \{\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12}\} \times \{\mathrm{Tr}_2\, F_2[(\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12})^{-1}\, \mathrm{Tr}_1(E_1 \otimes 1)\rho_{12}]\}.$$

Hence,

$$\mathrm{Tr}_{12}(E_1 \otimes F_2)\rho_{12} = \{\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12}\} \times \{\mathrm{Tr}_2\, F_2 \rho_2^{(c)}\} \tag{A.1}$$

where

$$\rho_2^{(c)} \equiv \{\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12}\}^{-1}\, \mathrm{Tr}_1(E_1 \otimes 1)\rho_{12}$$

can be viewed as the *conditional state* of subsystem 2 after the occurrence of the event $(E_1 \otimes 1)$ in the state ρ_{12}.

This interpretation is based on three facts:

(i) The operator $\rho_2^{(c)}$ is positive:

$$\forall |\psi\rangle_2 \in \mathscr{H}_2: \langle\psi|\rho_2^{(c)}|\psi\rangle_2 = \{\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12}\}^{-1}\, \mathrm{Tr}_{12}(E_1 \otimes |\psi\rangle_2\langle\psi|_2)\rho_{12} \geqslant 0$$

and obviously has trace equalling one.

(ii) The projector $F_2 \in \mathscr{P}(\mathscr{H}_2)$ is arbitrary, hence, as is well known, it determines (uniquely) the statistical operator $\rho_2^{(c)}$ through the standard quantum-mechanical probability prediction $\mathrm{Tr}_2\, F_2 \rho_2^{(c)}$ (the second factor on the RHS of (A.1)).

(iii) Viewing ρ_{12} as defined on a Boolean subalgebra of the quantum logic $\mathscr{P}(\mathscr{H}_{12})$, quantum-mechanical prediction reduces, as is well known, to classical probability. Then, $\mathrm{Tr}_2\, F_2 \rho_2^{(c)}$ can be interpreted as the definition of conditional probability as usual in classical probability theory.

Coming back to state-dependent implication, the relation '$E_1 \leqslant_{\rho_{12}} F_2$', in view of (5) and (A.1), amounts to

$$\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12} \times \mathrm{Tr}_2\, F_2 \rho_2^{(c)} = \mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12}$$

i.e. putting aside the trivial case $\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12} = 0$, to

$$\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12} > 0 \qquad \mathrm{Tr}_2\, F_2 \rho_2^{(c)} = 1.$$

In words: the relation '$E_1 \leqslant_{\rho_{12}} F_2$', for a given first-subsystem event $(E_1 \otimes 1)$ having a positive probability in the given composite-system state ρ_{12}, amounts to the fact that the given second-subsystem event F_2 is *statistically certain in the conditional state* that comes about as a consequence of the occurrence of $(E_1 \otimes 1)$ in ρ_{12}. This is so in any measurement of this event. (More about this in Herbut 1986, subsection 2.1.) Hence, this is checkable in the laboratory.

Illustration. We take the well known Bohm case of an Einstein–Podolsky–Rosen two-particle spin state (Bohm 1952, Einstein *et al* 1935). After having abstracted away the spatial degrees of freedom, one has:

$$|\Phi\rangle_{12} \equiv (1/2)^{1/2}(|+, \boldsymbol{u}\rangle_1 |-, \boldsymbol{u}\rangle_2 - |-, \boldsymbol{u}\rangle_1 |+, \boldsymbol{u}\rangle_2)$$

where $\boldsymbol{u}$ is an arbitrary unit vector in $\boldsymbol{R}_3$ (the spin up and down states are taken along it). It is well known that however one chooses $\boldsymbol{u}$, the given state vector $|\Phi\rangle_{12}$ has one and the same form. (On this fact is based the Einstein–Podolsky–Rosen paradox. We will not discuss it here.)

It is easy to see that taking, for example, $E_1 \equiv |+, \boldsymbol{u}\rangle_1\langle +, \boldsymbol{u}|_1$, $F_2 \equiv |-, \boldsymbol{u}\rangle_2\langle -, \boldsymbol{u}|_2$, $\rho_{12} \equiv |\Phi\rangle_{12}\langle\Phi|_{12}$, one has state-dependent implication $E_1 \leqslant_{\rho_{12}} F_2$.

Thus, there is sufficient physical motivation for a detailed mathematical investigation of state-dependent implication.

Appendix 2

If $E, F, G \in \mathscr{P}(\mathscr{H})$, $E \leqslant_\rho F$, $F \leqslant_\rho G$, and $EG \neq GE$, then it does not necessarily follow that $E \leqslant_\rho G$.

Proof. We take the more restricted definition (4) of '$\leqslant_\rho$', which boils down to

$$0 < \mathrm{Tr}\, E\rho = \mathrm{Tr}\, EFE\rho. \tag{A.2}$$

Consider a Hilbert space of at least five dimensions with the following orthonormal (sub)basis in it:

$$\{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle\}.$$

We define:

$$E \equiv |1\rangle\langle 1| + |3\rangle\langle 3| \qquad F \equiv |1\rangle\langle 1| + |2\rangle\langle 2|$$

$$G \equiv |g\rangle\langle g| + |g'\rangle\langle g'|$$

where

$$|g\rangle \equiv 2^{-1/2}(|1\rangle + |2\rangle) \qquad |g'\rangle \equiv 2^{-1/2}(|3\rangle + |4\rangle)$$

finally, $\rho \equiv |\psi\rangle\langle\psi|$, where

$$|\psi\rangle \equiv 2^{-1/2}(|g\rangle + |5\rangle).$$

In order to test the validity of (A.2), we evaluate:

$$\mathrm{Tr}\, E\rho = \langle\psi|E|\psi\rangle = \|\langle\psi|1\rangle\|^2 = \|2^{-1/2}2^{-1/2}\|^2 = \tfrac{1}{4}$$

$$\mathrm{Tr}\, F\rho = \langle\psi|F|\psi\rangle = \|F|\psi\rangle\|^2 = \|2^{-1/2}|g\rangle\|^2 = \tfrac{1}{2}$$

$$\mathrm{Tr}\, G\rho = \langle\psi|G|\psi\rangle = \|2^{-1/2}|g\rangle\|^2 = \tfrac{1}{2}$$

$$\mathrm{Tr}\, EFE\rho = \langle\psi|EFE|\psi\rangle = \langle\psi|2^{-1/2}2^{-1/2}|1\rangle = 2^{-1}2^{-1/2}2^{-1/2} = \tfrac{1}{4}.$$

So far we can infer from (A.2) that $E \leqslant_\rho F$, as claimed. Further,

$$\mathrm{Tr}\, FGF\rho = \langle\psi|FGF|\psi\rangle = 2^{-1/2}\langle\psi|g\rangle = 2^{-1/2}2^{-1/2} = \tfrac{1}{2}.$$

Hence, also $F \leqslant_\rho G$, as claimed. Finally,

$$\text{Tr}\, EGE\rho = \langle\psi|EGE|\psi\rangle = 2^{-1/2}2^{-1/2}2^{-1/2}2^{-1/2}\langle 1|G|1\rangle$$
$$= \tfrac{1}{4}|\langle 1|g\rangle|^2 = \tfrac{1}{4}\tfrac{1}{2} = \tfrac{1}{8}.$$

Thus, it is not true that $E \leqslant_\rho G$. □

Appendix 3

Let us first establish a few, perhaps not so well known, auxiliary criteria for statistically certain events.

Lemma A.1. Let ρ be a state, and $\rho = \Sigma_i w_i |i\rangle\langle i|$ ($\forall i$: $w_i > 0$) an arbitrary decomposition of ρ into pure states. Then an event E is certain in ρ if and only if it is so in every pure state $|i\rangle$.

Proof. Sufficiency is obvious. To prove necessity, one may argue as follows:

$\text{Tr}\, E\rho = 1 \Rightarrow \Sigma_i w_i \,\text{Tr}\, E|i\rangle\langle i| = \Sigma_i w_i \Rightarrow \Sigma_i w_i (1 - \langle i|E|i\rangle) = 0 \Rightarrow \forall i: \quad \langle i|E|i\rangle = 1.$ □

Lemma A.2. An event E is certain in a state ρ if and only if $E\rho = \rho$.

Proof. Sufficiency is obvious. To show that the condition is also necessary, we take a decomposition $\rho = \Sigma_i w_i |i\rangle\langle i|$ (e.g. a spectral form of ρ). According to lemma A.1, we have $\forall i$: $\langle i|E|i\rangle = 1$. This implies for each value of i:

$\langle i|E^\perp|i\rangle = 0 \Rightarrow \|E^\perp|i\rangle\|^2 = 0 \Rightarrow E|i\rangle = |i\rangle \Rightarrow E|i\rangle\langle i| = |i\rangle\langle i| \Rightarrow E\rho = \rho.$ □

Lemma A.3. An event E is certain in a state ρ if and only if $E\rho E = \rho$.

Proof. Sufficiency is a consequence of commutation under the trace. By utilizing lemma A.2, necessity is seen to hold as follows:

$\text{Tr}\, E\rho = 1 \Rightarrow E\rho = \rho \Rightarrow$ (as seen by adjoining) $\rho E = \rho \Rightarrow E\rho E = \rho.$ □

Lemma A.4. Two commuting projectors E, F and a statistical operator ρ stand in the relation $\text{Tr}\, E\rho = \text{Tr}\, EF\rho$ (cf (5)) if and only if

$$E\rho E = FE\rho EF \tag{A.3}$$

is valid.

Proof. Sufficiency is seen by taking the trace of (A.3) and by making use of commutation under the trace, of idempotency, and of commutation of E and F.

Necessity for the $\text{Tr}\, E\rho > 0$ case follows from relation (4) (equivalent to (5) in this case) and lemma A.3. If If $\text{Tr}\, E\rho = 0$, then also $\text{Tr}\, E\rho E = 0$, and $E\rho E = 0$ (because no positive operator other than zero has zero trace). Then (A.3) is trivially satisfied. □

Now we come to the main result of this appendix.

Lemma A.5. If $E, F, G \in \mathscr{B}$ $(\subset \mathscr{P}(\mathscr{H}))$, where $\mathscr{B}$ is a Boolean subalgebra, and $E \leqslant_\rho F$, $F \leqslant_\rho G$, then necessarily $E \leqslant_\rho G$.

Proof. We make use of definition 1 and (5), the latter in its equivalent form $E\rho E = FE\rho EF$ (cf lemma A.4). By assumption, we have

$$E\rho E = FE\rho EF \qquad \text{and} \qquad F\rho F = GF\rho FG.$$

Then

$$E\rho E = EF\rho FE = EGF\rho FGE = GFE\rho EFG = GE\rho EG. \qquad \square$$

Appendix 4

An event E is 'impossible' in a state ρ if and only if $E\rho = 0$.

Proof. According to lemma A.2, one has

$$\mathrm{Tr}\, E^\perp \rho = 1 \Leftrightarrow E^\perp \rho = \rho \Leftrightarrow E\rho = 0. \qquad \square$$

Appendix 5

Let ρ be an arbitrary given statistical operator, and $\mathscr{B}$ an arbitrary given Boolean subalgebra of $\mathscr{P}(\mathscr{H})$. Then $\mathscr{B}_0$ $(\equiv \{G: G \in \mathscr{B}, \mathrm{Tr}\, G\rho = 0\})$ and $\mathscr{B}_1$ $(\equiv \{G: G \in \mathscr{B}, \mathrm{Tr}\, G\rho = 1\})$ are equivalence classes with respect to $\sim_\rho$ (cf definition 2).

Proof. If $E \in \mathscr{B}_0$, and $F \in \mathscr{B}$, then (5) is satisfied because, according to appendix 4, $\mathrm{Tr}\, E\rho = 0$ implies $E\rho = 0$, hence $\mathrm{Tr}\, EF\rho = \mathrm{Tr}\, FE\rho = 0$. Thus, $E \leqslant_\rho F$ (cf definition 1). In particular, if $E, F \in \mathscr{B}_0$, then $E \sim_\rho F$. Conversely, if $E \leqslant_\rho F$, and $F \in \mathscr{B}_0$, then also $E \in \mathscr{B}_0$, because (5) reads $\mathrm{Tr}\, EF\rho = \mathrm{Tr}\, E\rho$, and $\mathrm{Tr}\, F\rho = 0$ implies $F\rho = 0$, and hence $\mathrm{Tr}\, EF\rho = \mathrm{Tr}\, E = 0$. In particular, if $E \sim_\rho F$, and $F \in \mathscr{B}_0$, then also $E \in \mathscr{B}_0$. Thus, $\mathscr{B}_0$ is an equivalence class with respect to '$\sim_\rho$'.

If $E \in \mathscr{B}$, $F \in \mathscr{B}_1$, then $E \leqslant_\rho F$, because $\mathrm{Tr}\, F\rho = 1$ implies (cf lemma A.2) $F\rho = \rho$, hence $\mathrm{Tr}\, EF\rho = \mathrm{Tr}\, E\rho$ (cf definition 1 and (5)). In particular, if $E, F \in \mathscr{B}_1$, then $E \sim_\rho F$ (cf definition 2).

Let $E \sim_\rho F$, and $E \in \mathscr{B}_1$, $F \in \mathscr{B}$. Then the relation $E \leqslant_\rho F$ (4) and lemma A.2 give $F(E\rho E) = E\rho E$. Further, on account of lemma A.3, $E\rho E = \rho$. Hence, $F\rho = \rho$, i.e. $F \in \mathscr{B}_1$. Thus, $\mathscr{B}_1$ is an equivalence class with respect to '$\sim_\rho$'. $\square$

Appendix 6. Proof of relation (7)

$$[0] = \mathscr{B}_0 = \{E: E \leqslant Q_0\}$$

where Q_0 is the null projector (or $Q_0^\perp$ is the range projector) of ρ.

Let $\rho = \Sigma_i r_i |i\rangle\langle i|$ be a spectral form of ρ with positive characteristic values r_i. By assumption $\mathrm{Tr}\, E\rho = 0$. Hence, $\mathrm{Tr}\, E^\perp \rho = 1$, and, on account of lemma A.2, $E^\perp \rho = \rho$. Further, owing to lemma A.1, $\forall i: E^\perp |i\rangle = |i\rangle$, or $E|i\rangle = 0$. Since $Q_0^\perp = \Sigma_i |i\rangle\langle i|$, $EQ_0^\perp = 0$, or $EQ_0 = E$, i.e. $E \leqslant Q_0$. $\square$

Appendix 7. Proof of theorem 3

Sufficiency. We assume that the preorder '$\rightarrow$' in $\mathscr{B}$ satisfies the three relations given in the theorem. First, we have to show that

$$\Delta \equiv \{F: 0 \rightarrow F, F \rightarrow 0\}$$

is an ideal of $\mathscr{B}$ (see section 2).

Let $E, F \in \Delta$. Then $E \rightarrow 0$ and $F \rightarrow 0$, hence relation (iii)′ (cf lemma 3) implies: $E \vee F \rightarrow 0$. On the other hand, $0 \rightarrow E \vee F$ as a consequence of $0 \leqslant E \vee F$ (0 implies every element of $\mathscr{B}$) and of relation (i). Thus, $E \vee F \in \Delta$.

Let $E \in \mathscr{B}$, $F \in \Delta$, and $E \leqslant F$. Then $E \rightarrow F$ (relation (i)). Since also $F \rightarrow 0$, transitivity of '$\rightarrow$' gives $E \rightarrow 0$. Since also $0 \rightarrow E$ (consequence of $0 \leqslant E$ and of relation (i)), $E \in \Delta$. Thus, Δ is an ideal as claimed.

Next, we have to prove that both $E \rightarrow F$ and $F \rightarrow E$ are valid in $\mathscr{B}$ if and only if $EF^{\perp}$, $E^{\perp}F \in \Delta$.

We assume $E \rightarrow F$ and $F \rightarrow E$. One has $EF^{\perp} = (E \wedge F^{\perp}) \leqslant F^{\perp}$ implying, on account of relation (i), $EF^{\perp} \rightarrow F^{\perp}$. Analogously, $EF^{\perp} \rightarrow E$. Now, $E \rightarrow F$ and transitivity of '$\rightarrow$' give $EF^{\perp} \rightarrow F$. Then relation (iii) entails $EF^{\perp} \rightarrow (F^{\perp} \wedge F) = 0$. Since also $0 \rightarrow EF^{\perp}$ (due to $0 \leqslant EF^{\perp}$), we have $EF^{\perp} \in \Delta$. Symmetrically, $FE^{\perp} \in \Delta$. Thus, '$\sim_{\rightarrow}$' implies '$\sim_{\Delta}$'.

We assume that $EF^{\perp}$, $E^{\perp}F \in \Delta$. Evidently,

$$E = EF^{\perp} + EF = EF^{\perp} \vee EF. \tag{A.4}$$

Further, $EF^{\perp} \rightarrow 0 \rightarrow E^{\perp}F \rightarrow F$. (The last step is due to $E^{\perp}F \leqslant F$ and to relation (i).) Transitivity of '$\rightarrow$' implies

$$EF^{\perp} \rightarrow F. \tag{A.5}$$

On account of $EF \leqslant F$ and (i), one has $EF \rightarrow F$. This relation and (A.5), due to (A.4) and relation (iii)′ from lemma 3, give $E \rightarrow F$. Symmetrically, one proves $F \rightarrow E$. Thus, '$\sim_{\Delta}$' implies '$\sim_{\rightarrow}$'. Altogether, the two equivalence relations coincide, as claimed.

Next we have to show that also the two induced order relations in $\mathscr{B}/\Delta$ $(=\mathscr{B}/\sim_{\rightarrow})$ coincide.

Let $[E] \leqslant [F]$ (elements of $\mathscr{B}/\Delta$). Then $[E] \wedge [F] = [E] = [EF]$, i.e. $E \sim_{\Delta} EF$. Hence, $E(EF)^{\perp} \in \Delta$, implying $E(EF)^{\perp} \rightarrow 0$. Further, $E(EF)^{\perp} = E(1 - EF) = EF^{\perp}$. Thus, $EF^{\perp} \rightarrow 0$. Since also $0 \rightarrow F$, transitivity entails $EF^{\perp} \rightarrow F$. Further, $EF \leqslant F$ gives $EF \rightarrow F$ (due to relation (i)). Finally, relation (iii)′ (from lemma 3) and (A.4) entail $E \rightarrow F$. Thus, $[E] \rightarrow [F]$, as claimed.

Let $[E] \rightarrow [F]$. Then $E \rightarrow F$. Since $E \rightarrow E$, and $EF = E \wedge F$, relation (iii) implies $E \rightarrow EF$. On the other hand, $EF \leqslant E$, and hence $EF \rightarrow E$ (relation (i)). Thus, $[E] = [EF] = [E \wedge F] = [E] \wedge [F]$, i.e. $[E] \leqslant [F]$, as claimed.

Necessity. We assume that '$\sim_{\rightarrow}$' and '$\sim_{\Delta}$' coincide, and $[E] \leqslant [F]$ if and only if $[E] \rightarrow [F]$.

Let $E \leqslant F$. Then $EF = E$, hence $[E] \wedge [F] = [E]$, and $[E] \leqslant [F]$. Then also $[E] \rightarrow [F]$, implying $E \rightarrow F$. Hence relation (i) is valid.

Let $E \rightarrow F$. Then $[E] \rightarrow [F]$, and also $[E] \leqslant [F]$. Hence $[F]^{\perp} \leqslant [E]^{\perp}$, i.e. $[F^{\perp}] \leqslant [E^{\perp}]$, and $[F^{\perp}] \rightarrow [E^{\perp}]$. Finally, $F^{\perp} \rightarrow E^{\perp}$. Thus, relation (ii) is valid.

Let $E \rightarrow F$ and $E \rightarrow G$. Then, $[E] \rightarrow [F]$ and $[E] \rightarrow [G]$. Hence $[E] \leqslant [F]$ and $[E] \leqslant [G]$, implying $[E] \leqslant ([F] \wedge [G]) = [FG]$. Then also $[E] \rightarrow [FG]$, which finally implies $E \rightarrow FG$. Thus, relation (iii) is valid. □

Appendix 8. Proof of theorem 4

If $E \leqslant F$, then $EF = E$, and $\mathrm{Tr}\, EF\rho = \mathrm{Tr}\, E\rho$. Hence, $E \leqslant_\rho F$ (cf definition 1 and (5)). Thus, relation (i) of theorem 3 is valid for '$\leqslant_\rho$'.

If $E \leqslant_\rho F$, then $\mathrm{Tr}\, E\rho = \mathrm{Tr}\, EF\rho$. Hence,

$$\mathrm{Tr}\, F^\perp \rho \equiv \mathrm{Tr}(1-F)\rho = \mathrm{Tr}(1-F-E+EF)\rho = \mathrm{Tr}(1-F)(1-E)\rho = \mathrm{Tr}\, F^\perp E^\perp \rho.$$

Thus, $F^\perp \leqslant_\rho E^\perp$, and also relation (ii) of theorem 3 holds for '$\leqslant_\rho$'.

Finally, we assume $E \leqslant_\rho F$ and $E \leqslant_\rho G$. Then, on account of lemma A.4, we have

$$FE\rho EF = E\rho E \qquad \text{and} \qquad GE\rho EG = E\rho E.$$

Further,

$$FGE\rho EFG = G(FE\rho EF)G = GE\rho EG = E\rho E.$$

Repeated use of lemma A.4 finally gives $E \leqslant_\rho FG$.

Thus, also relation (iii) of theorem 3 is valid for '$\leqslant_\rho$'. □

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